

# Regularity of solutions in semilinear elliptic theory

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## Abstract

We study the semilinear Poisson equation

$$\Delta u = f(x, u) \quad \text{in} \quad B_1. \quad (1)$$

Our main results provide conditions on  $f$  which ensure that weak solutions of (1) belong to  $C^{1,1}(B_{1/2})$ . In some configurations, the conditions are sharp.

## 1 Introduction

The semilinear Poisson equation (1) encodes stationary states of the nonlinear heat, wave, and Schrödinger equation. In the case when  $f$  is the Heaviside function in the  $u$ -variable, (1) reduces to the classical obstacle problem. For an introduction to classical semilinear theory, see [BS11, Caz06].

It is well-known that weak solutions of (1) belong to the usual Sobolev space  $W^{2,p}(B_{1/2})$  for any  $1 \leq p < \infty$  provided  $f \in L^\infty$ . Recent research activity has thus focused on identifying conditions on  $f$  which ensure  $W^{2,\infty}(B_{1/2})$  regularity of  $u$ .

### 1.1 The classical theory

There are simple examples which illustrate that continuity of  $f = f(x)$  does not necessarily imply that  $u$  has bounded second derivatives: for  $p \in (0, 1)$  and  $x \in \mathbb{R}^2$  such that  $|x| < 1$ , the function

$$u(x) = x_1 x_2 (-\log |x|)^p$$

has a continuous Laplacian but is not in  $C^{1,1}$  [Sha15]. However, if  $f$  is Hölder continuous, then it is well-known that  $u \in C^{2,\alpha}$ ; if  $f$  is Dini continuous, then  $u \in C^2$  [GT01, Kov99]. The sharp condition which guarantees bounded second derivatives of  $u$  is the  $C^{1,1}$  regularity of  $f * N$  where  $N$  is the Newtonian potential and  $*$  denotes convolution; this requirement is strictly weaker than Dini continuity of  $f$ .

In the general case, the state-of-the-art is a theorem of Shahgholian [Sha03] which states that  $u \in C^{1,1}$  whenever  $f = f(x, u)$  is Lipschitz in  $x$ , uniformly

in  $u$ , and  $\partial_u f \geq -C$  weakly for some  $C \in \mathbb{R}$ . In some configurations this illustrates regularity for continuous functions  $f = f(u)$  which are strictly below the classical Dini-threshold in the  $u$ -variable, e.g. the odd reflection of

$$f(t) = -\frac{1}{\log(t)}$$

about the origin. Shahgholian's theorem is proved via the celebrated Alt-Caffarelli-Friedman (ACF) monotonicity formula and it seems difficult to weaken the assumptions by this method. On the other hand, Koch and Nadirashvili [KN] recently constructed an example which illustrates that the continuity of  $f$  is not sufficient to deduce that weak solutions of  $\Delta u = f(u)$  are in  $C^{1,1}$ .

We say  $f = f(x, u)$  satisfies assumption **A** provided that  $f$  is Dini continuous in  $u$ , uniformly in  $x$ , and has a  $C^{1,1}$  Newtonian potential in  $x$ , uniformly in  $u$  (see §3). One of our main results is the following statement.

**Theorem 1.1.** *Suppose  $f$  satisfies assumption **A**. Then any solution of (1) is  $C^{1,1}$  in  $B_{1/2}$ .*

Our assumption includes functions which fail to satisfy both conditions in Shahgholian's theorem, e.g.

$$f(x_1, x_2, t) = \frac{x_1}{\log(|x_2|)(-\log|t|)^p},$$

for  $p > 1$ ,  $x = (x_1, x_2) \in B_1$  and  $t \in (-1, 1)$ . The Newtonian potential assumption in the  $x$ -variable is essentially sharp whereas the condition in the  $t$ -variable is in general not comparable with Shahgholian's assumption.

The proof of Theorem 1.1 does not invoke monotonicity formulas and is self-contained. We consider the  $L^2$  projection of  $D^2 u$  on the space of Hessians generated by second order homogeneous harmonic polynomials on balls with radius  $r > 0$  and show that the projections stay uniformly bounded as  $r \rightarrow 0^+$ . Although this approach has proven effective in dealing with a variety of free boundary problems [ALS13, FS14, IM15, IM], Theorem 1.1 illustrates that it is also useful in extending and refining the classical elliptic theory.

## 1.2 Singular case: the free boundary theory

In §4 we study the PDE (1) for functions  $f = f(x, u)$  which are discontinuous in the  $u$ -variable at the origin.

If the discontinuity of  $f$  is a jump discontinuity, (1) has the structure

$$f(x, u) = g_1(x, u)\chi_{\{u>0\}} + g_2(x, u)\chi_{\{u<0\}}, \quad (2)$$

where  $g_1, g_2$  are continuous functions such that

$$g_1(x, 0) \neq g_2(x, 0), \quad \forall x \in B_1,$$

and  $\chi_\Omega$  defines the indicator function of the set  $\Omega$ .

Our aim is to find the most general class of coefficients  $g_i$  which generate interior  $C^{1,1}$  regularity.

The classical obstacle problem is obtained by letting  $g_1 = 1, g_2 = 0$ , and it is well-known that solutions have second derivatives in  $L^\infty$  [PSU12]. Nevertheless, by selecting  $g_1 = -1, g_2 = 0$ , one obtains the so-called unstable obstacle problem. Elliptic theory and the Sobolev embedding theorem imply that any weak solution belongs to  $C^{1,\alpha}$  for any  $0 < \alpha < 1$ . It turns out that this is the best one can hope for: there exists a solution which fails to be in  $C^{1,1}$  [AW06]. Hence, if there is a jump at the origin,  $C^{1,1}$  regularity can hold only if the jump is positive and this gives rise to:

**Assumption B.**  $g_1(x, 0) - g_2(x, 0) \geq \sigma_0$ ,  $x \in B_1$  for some  $\sigma_0 > 0$ .

The free boundary  $\Gamma = \partial\{u \neq 0\}$  consists of two parts:  $\Gamma^0 = \Gamma \cap \{\nabla u = 0\}$  and  $\Gamma^1 = \Gamma \cap \{\nabla u \neq 0\}$ . The main difficulty in proving  $C^{1,1}$  regularity is the analysis of points where the gradient of the function vanishes. In this direction we establish the following result.

**Theorem 1.2.** *Suppose  $g_1, g_2$  satisfy **A** and **B**. Then if  $u$  is a solution of (1),  $\|u\|_{C^{1,1}(K)} < \infty$  for any  $K \Subset B_{1/2}(0) \setminus \bar{\Gamma}^1$ .*

At points where the gradient does not vanish, the implicit function theorem yields that the free boundary is locally a  $C^{1,\alpha}$  graph for any  $0 < \alpha < 1$ . The solution  $u$  changes sign across the free boundary, hence it locally solves the equation  $\Delta u = g_1(x, u)$  on the side where it is positive and  $\Delta u = g_2(x, u)$  on the side where it is negative. If the coefficients  $g_i$  are regular enough to provide  $C^{1,1}$  solutions up to the boundary – this is encoded in assumption **C**, see §4 – then we obtain full  $C^{1,1}$  regularity.

**Theorem 1.3.** *Suppose  $g_1, g_2$  satisfy **A**, **B** and **C**. Let  $u$  be a solution of (1) and  $0 \in \Gamma^0$ . Then  $u \in C^{1,1}(B_{\rho_0}(0))$ , for some  $\rho_0 > 0$ .*

Equation (1) with right-hand side of the form (2) is a generalization of the well-studied two-phase membrane problem, where  $g_i(x, u) = \lambda_i(x)$ ,  $i = 1, 2$ . The  $C^{1,1}$  regularity in the case when  $\lambda_1 \geq 0, \lambda_2 \leq 0$  are two constants satisfying **B** was obtained by Uraltseva [Ura01] via the ACF monotonicity formula. Moreover, Shahgholian proved this result for Lipschitz coefficients which satisfy **B** [Sha03, Example 2]. If the coefficients are Hölder continuous, the ACF method does not directly apply and under the stronger assumption that  $\inf \lambda_1 > 0$  and  $\inf -\lambda_2 > 0$ , Edquist, Lindgren, Shahgholian [LSE09] obtained the  $C^{1,1}$  regularity via an analysis of blow-up limits and a classification of global solutions (see also [LSE09, Remark 1.3]). Theorem 1.3 improves and extends this result.

The difficulty in the case when  $g_i$  depend also on  $u$  is that if  $v := u + L$  for some linear function  $L$ , then  $v$  is no longer a solution to the same equation, so one has to get around the lack of linear invariance. Our technique exploits that linear perturbations do not affect certain  $L^2$  projections.

The proof of Theorem 1.3 does not rely on classical monotonicity formulas or classification of global solutions. Rather, our method is based on an identity

which provides monotonicity in  $r$  of the square of the  $L^2$  norm of the projection of  $u$  onto the space of second order homogeneous harmonic polynomials on the sphere of radius  $r$ .

Theorems 1.2 & 1.3 deal with the case when  $f$  has a jump discontinuity. If  $f$  has a removable discontinuity, (1) has the structure

$$\Delta u = g(x, u)\chi_{u \neq 0}. \quad (3)$$

In this case, one may merge some observations in the proofs of the previous results with the method in [ALS13] and prove the following theorem.

**Theorem 1.4.** *If  $g$  satisfies assumption **A**, then every solution of (3) is in  $C^{1,1}(B_{1/2})$ .*

Theorems 1.1 - 1.4 provide a comprehensive theory for the general semilinear Poisson equation where the free boundary theory is encoded in the regularity assumption of  $f$  in the  $u$ -variable.

## 2 Technical tools

Throughout the text, the right-hand side of (1) is assumed to be bounded. Moreover,  $\mathcal{P}_2$  denotes the space of second order homogeneous harmonic polynomials. A useful elementary fact is that all norms on  $\mathcal{P}_2$  are equivalent.

**Lemma 2.1.** *The space  $\mathcal{P}_2$  is a finite dimensional linear space. Consequently, all norms on  $\mathcal{P}_2$  are equivalent.*

For  $u \in W^{2,2}(B_1)$ ,  $y \in B_1$  and  $r \in (0, \text{dist}(y, \partial B_1))$ ,  $\Pi_y(u, r)$  is defined to be the  $L^2$  projection operator on  $\mathcal{P}_2$  given by

$$\inf_{h \in \mathcal{P}_2} \int_{B_1} \left| D^2 \frac{u(rx + y)}{r^2} - D^2 h \right|^2 dx = \int_{B_1} \left| D^2 \frac{u(rx + y)}{r^2} - D^2 \Pi_y(u, r) \right|^2 dx.$$

Calderon-Zygmund theory yields the following useful inequality for re-scalings of weak solutions of (1).

**Lemma 2.2.** *Let  $u$  solve (1),  $y \in B_{1/2}$ , and  $r \leq 1/4$ . Then for*

$$\tilde{u}_r(x) = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2}$$

*it follows that for  $1 \leq p < \infty$  and  $0 < \alpha < 1$ ,*

$$\|\tilde{u}_r - \Pi_y(u, r)\|_{W^{2,p}(B_1)} \leq C(n, \|f\|_{L^\infty(B_1 \times \mathbb{R})}, \|u\|_{L^\infty(B_1)}, p),$$

*and*

$$\|\tilde{u}_r - \Pi_y(u, r)\|_{C^{1,\alpha}(B_1)} \leq C(n, \|f\|_{L^\infty(B_1 \times \mathbb{R})}, \|u\|_{L^\infty(B_1)}, \alpha).$$

*Proof.* By Calderon-Zygmund theory (e.g. [ALS13, Theorem 2.2]),

$$\|D^2u\|_{BMO(B_{1/2})} \leq C;$$

in particular,

$$\int_{B_{3/2}} |D^2\tilde{u}_r - \overline{D^2\tilde{u}_r}|^2 \leq C,$$

where  $\overline{D^2\tilde{u}_r}$  is the average of  $D^2\tilde{u}_r$  on  $B_{3/2}$ . Now let

$$a = a(f, r, y) = \oint_{B_{3/2}} f(rx + y, u(rx + y)) dx$$

and note that this quantity is uniformly controlled by  $\|f\|_{L^\infty(B_1 \times \mathbb{R})}$ ; this fact, and the definition of  $\Pi$  yields (note:  $\text{trace}(\overline{D^2u} - \frac{a}{n}Id) = 0$ ),

$$\int_{B_{3/2}} |D^2(\tilde{u}_r - \Pi_0(\tilde{u}_r, 3/2))|^2 \leq \int_{B_{3/2}} |D^2\tilde{u}_r - (\overline{D^2u} - \frac{a}{n}Id)|^2 \leq C_1.$$

Two applications of Poincaré's inequality together with the above estimate implies

$$\|\tilde{u}_r - \Pi_y(u, r) - \overline{\nabla\tilde{u}_r} \cdot x - \overline{\tilde{u}_r}\|_{W^{2,2}(B_{3/2})} \leq C_2,$$

where the averages are taken over  $B_{3/2}$ . Elliptic theory (e.g. [GT01, Theorem 9.1]) yields that for any  $1 \leq p < \infty$ ,

$$\|\tilde{u}_r - \Pi_y(u, r) - \overline{\nabla\tilde{u}_r} \cdot x - \overline{\tilde{u}_r}\|_{W^{2,p}(B_{3/2})} \leq C_3.$$

Let  $\phi := \tilde{u}_r - \overline{\nabla\tilde{u}_r} \cdot x - \overline{\tilde{u}_r}$ . We have that  $\phi(0) = -\overline{\tilde{u}_r}$  and  $\nabla\phi(0) = -\overline{\nabla\tilde{u}_r}$ ; however, by the Sobolev embedding theorem,  $\phi$  is  $C^{1,\alpha}$  and thus

$$|\phi(0)| + |\nabla\phi(0)| \leq C_4$$

completing the proof of the  $W^{2,p}$  estimate. The  $C^{1,\alpha}$  estimate likewise follows from the Sobolev embedding theorem.  $\square$

Our analysis requires several additional simple technical lemmas involving the projection operator.

**Lemma 2.3.** *For any  $u \in W^{2,2}(B_1)$  and  $s \in [1/2, 1]$ ,*

$$\|\Pi_0(u, s) - \Pi_0(u, 1)\|_{L^2(B_1)} \leq C\|\Delta u\|_{L^2(B_1)},$$

and

$$\|\Pi_0(u, s) - \Pi_0(u, 1)\|_{L^\infty(B_1)} \leq C\|\Delta u\|_{L^2(B_1)},$$

for some constant  $C = C(n)$ .

*Proof.* Let  $f = \Delta u$  and  $v$  be the Newtonian potential of  $f$ , i.e.

$$v(x) = \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)\chi_{B_1}(y)}{|x-y|^{n-2}} dx,$$

where  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . Since  $u - v$  is harmonic,

$$\Pi_0(u - v, s) = \Pi_0(u - v, 1);$$

therefore

$$\Pi_0(u, s) - \Pi_0(u, 1) = \Pi_0(v, s) - \Pi_0(v, 1).$$

Invoking bounds on the projection (e.g. [ALS13, Lemma 3.2]) and Calderon-Zygmund theory (e.g. [ALS13, Theorem 2.2]), it follows that

$$\begin{aligned} \|\Pi_0(u, s) - \Pi_0(u, 1)\|_{L^2(B_1)} &= \|\Pi_0(v, s) - \Pi_0(v, 1)\|_{L^2(B_1)} \\ &\leq C\|\Delta v\|_{L^2(B_1)} = C\|\Delta u\|_{L^2(B_1)}. \end{aligned}$$

The  $L^\infty$  bound follows from the equivalence of the norms in the space  $\mathcal{P}_2$ .  $\square$

**Lemma 2.4.** *Let  $u$  solve (1). Then for all  $0 < r \leq 1/4$ ,  $s \in [1/2, 1]$  and  $y \in B_{1/2}$ ,*

$$\sup_{B_1} |\Pi_y(u, rs) - \Pi_y(u, r)| \leq C,$$

and

$$\sup_{B_1} |\Pi_y(u, r)| \leq C \log(1/r),$$

for some constant  $C = C(n, \|f\|_{L^\infty(B_1 \times \mathbb{R})}, \|u\|_{L^\infty(B_1)})$ .

*Proof.* Note that

$$\Pi_y(u, rs) - \Pi_y(u, r) = \Pi_0(\tilde{u}_r, s) - \Pi_0(\tilde{u}_r, 1),$$

where

$$\tilde{u}_r(x) = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2}$$

as before. From Lemma 2.3 we have that

$$\|\Pi_0(\tilde{u}_r, s) - \Pi_0(\tilde{u}_r, 1)\|_{L^\infty(B_1 \times \mathbb{R})} \leq C\|\Delta \tilde{u}_r\|_{L^2(B_1)} \leq C\|f\|_{L^\infty(B_1)}.$$

As for the second inequality in the statement of the lemma let  $r_0 = 1/4$  and  $s \in [1/2, 1]$ . Then we have that

$$\begin{aligned} \sup_{B_1} |\Pi_y(u, sr_0/2^j)| &\leq \sup_{B_1} |\Pi_y(u, sr_0/2^j) - \Pi_y(u, r_0/2^j)| \\ &\quad + \sum_{k=0}^{j-1} \sup_{B_1} |\Pi_y(u, r/2^{k+1}) - \Pi_y(u, r/2^k)| \\ &\quad + \sup_{B_1} |\Pi_y(u, r_0)| \leq Cj \leq C \log\left(\frac{2^j}{sr_0}\right), \end{aligned}$$

for all  $j \geq 1$ .  $\square$

The previous tools imply a growth estimate on weak solutions solution of (1).

**Lemma 2.5.** *Let  $u$  solve (1). Then for  $y \in B_{1/2}$  and  $r > 0$  small enough,*

$$\sup_{B_r(y)} |u(x) - u(y) - (x - y) \nabla u(y)| \leq Cr^2 \log(1/r).$$

*Proof.* Let

$$\tilde{u}_r = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2}.$$

The assertion of the Lemma is equivalent to the estimate

$$\|\tilde{u}_r\|_{L^\infty(B_1)} \leq C \log(1/r),$$

for  $r$  small enough. Lemma 2.4 and the  $C^{1,\alpha}$  estimates of Lemma 2.2 imply

$$\begin{aligned} \|\tilde{u}_r\|_{L^\infty(B_1)} &\leq \|\tilde{u}_r - \Pi_y(u, r)\|_{L^\infty(B_1)} + \|\Pi_y(u, r)\|_{L^\infty(B_1)} \\ &\leq C + C \log(1/r) \leq C \log(1/r), \end{aligned}$$

provided  $r$  is small enough.  $\square$

Next lemma relates the boundedness of the projection operator and the boundedness of second derivatives of weak solutions of (1).

**Lemma 2.6.** *Let  $u$  be a solution to (1). If for each  $y \in B_{1/2}$  there is a sequence  $r_j(y) \rightarrow 0^+$  as  $j \rightarrow \infty$  such that*

$$M := \sup_{y \in B_{1/2}} \sup_{j \in \mathbb{N}} \|D^2 \Pi_y(u, r_j(y))\|_{L^\infty(B_{1/2})} < \infty,$$

*then*

$$|D^2 u| \leq C \quad \text{a.e. in } B_{1/2},$$

*for some constant  $C = C(M, n, \|f\|_{L^\infty(B_1 \times \mathbb{R})}, \|u\|_{L^\infty(B_1)}) > 0$ .*

*Proof.* Let  $y \in B_{1/2}$  be a Lebesgue point for  $D^2 u$  and  $r_j = r_j(y) \rightarrow 0^+$  as  $j \rightarrow \infty$ . Then by utilizing Lemma 2.2,

$$\begin{aligned} |D^2 u(y)| &= \lim_{j \rightarrow \infty} \int_{B_{r_j}(y)} |D^2 u(z)| dz \\ &\leq \limsup_{j \rightarrow \infty} \int_{B_{r_j}(y)} |D^2 u(z) - D^2 \Pi_y(u, r_j)| dz + M \\ &\leq C. \end{aligned}$$

Since a.e.  $z \in B_{1/2}$  is a Lebesgue point for  $D^2 u$ , the proof is complete.  $\square$

Next, we introduce another projection that we need for our analysis. Define  $Q_y(u, r)$  to be the minimizer of

$$\inf_{q \in \mathcal{P}_2} \int_{\partial B_1} \left| \frac{u(rx + y)}{r^2} - q(x) \right|^2 d\mathcal{H}^{n-1}.$$

The following lemma records the basic properties enjoyed by this projection, cf. [ALS13, Lemma 3.2].

**Lemma 2.7.** *i.  $Q_y(\cdot, r)$  is linear;*

*ii. if  $u$  is harmonic  $Q_y(u, s) = Q_y(u, r)$  for all  $s < r$ ;*

*iii. if  $u$  is a linear function then  $Q_y(u, r) = 0$ ;*

*iv. if  $u$  is a second order homogeneous polynomial then  $Q_y(u, r) = u$ ;*

*v.  $\|Q_0(u, s) - Q_0(u, 1)\|_{L^2(\partial B_1)} \leq C_s \|\Delta u\|_{L^2(B_1)}$ , for  $0 < s < 1$ ;*

*vi.  $\|Q_0(u, 1)\|_{L^2(\partial B_1)} \leq \|u\|_{L^2(\partial B_1)}$ .*

*Proof.* i. This is evident.

ii. It suffices to prove  $Q_y(u, r) = Q_y(u, 1)$  for  $r < 1$ . Let

$$\sigma_2 = \frac{Q_y(u, 1)}{\|Q_y(u, 1)\|_{L^2(\partial B_1)}}$$

and for  $i \neq 2$ , let  $\sigma_i$  be an  $i^{th}$  degree harmonic polynomial. Then there exist coefficients  $a_i$  such that

$$u(x + y) = \sum_{i=0}^{\infty} a_i \sigma_i(x), \quad x \in \partial B_1;$$

in particular,  $a_2 = \|Q_y(u, 1)\|$ . Let

$$v(x) = \sum_{i=0}^{\infty} a_i \sigma_i(x), \quad x \in B_1.$$

Then  $v$  is a harmonic and  $u(x + y) = v(x)$  for  $x \in \partial B_1$ . Hence, we have that  $u(x + y) = v(x)$  for  $x \in B_1$  and in particular

$$u(x + y) = \sum_{i=0}^{\infty} a_i \sigma_i(x), \quad x \in B_1.$$

Therefore

$$\frac{u(rx + y)}{r^2} = \sum_{i=0}^{\infty} a_i \frac{\sigma_i(rx)}{r^2} = \sum_{i=0}^{\infty} a_i r^{i-2} \sigma_i(x), \quad x \in B_1,$$

so  $Q_y(u, r) = a_2 \sigma_2(x) = Q_y(u, 1)$ .

iii. & iv. These are evident.

v. Similar to Lemma 2.3.

vi. This follows from the fact that  $Q_0(u, 1)$  is the  $L^2$  projection of  $u$ .

□



Next we prove some technical results for  $Q_y(u, r)$  and establish a precise connection between  $\Pi_y(u, r)$  and  $Q_y(u, r)$  by showing that the difference is uniformly bounded in  $r$ .

**Lemma 2.8.** *For  $u \in W^{2,p}(B_1(y))$  with  $p$  large enough and  $r \in (0, 1]$ ,*

$$\frac{d}{dr}Q_y(u, r) = \frac{1}{r}Q_0(x \cdot \nabla u(x+y) - 2u(x+y), r).$$

*Proof.* Firstly,

$$Q_y(u, r) = Q_0\left(\frac{u(rx+y)}{r^2}, 1\right).$$

Since  $u$  is  $C^{1,\alpha}$  if  $p$  large enough and  $Q$  is linear bounded operator, it follows that

$$\begin{aligned} \frac{d}{dr}Q_y(u, r) &= Q_0\left(\frac{d}{dr}\frac{u(rx+y)}{r^2}, 1\right) = Q_0\left(\frac{rx \cdot \nabla u(rx+y) - 2u(rx+y)}{r^3}, 1\right) \\ &= \frac{1}{r}Q_0(x \cdot \nabla u(x+y) - 2u(x+y), r). \end{aligned}$$

□

**Lemma 2.9.** *Let  $u \in W^{2,p}(B_1(y))$  with  $p$  large enough and  $q \in \mathcal{P}_2$ . Then*

$$\int_{B_1} q(x) \Delta u(x+y) dx = \int_{\partial B_1} q(x) (x \cdot \nabla u(x+y) - 2u(x+y)) d\mathcal{H}^{n-1}. \quad (4)$$

*Proof.* Integration by parts implies

$$\int_{B_1} q(x) \Delta u(x+y) dx = \int_{B_1} \Delta q(x) u(x+y) dx + \int_{\partial B_1} q(x) \frac{\partial u(x+y)}{\partial n} - u(x+y) \frac{\partial q(x)}{\partial n} d\mathcal{H}^{n-1}.$$

By taking into account that  $q$  is a second order homogeneous polynomial it follows that

$$\frac{\partial q(x)}{\partial n} = 2q(x), \quad x \in \partial B_1.$$

Moreover,

$$\frac{\partial u(x+y)}{\partial n} = x \cdot \nabla u(x+y), \quad x \in \partial B_1.$$

Combining these equations yields (4). □

**Lemma 2.10.** *Let  $u \in W^{2,p}(B_1(y))$  with  $p$  large enough and  $0 < r \leq 1$ . Then for every  $q \in \mathcal{P}_2$ ,*

$$\int_{\partial B_1} q(x) \frac{d}{dr}Q_y(u, r)(x) d\mathcal{H}^{n-1} = \frac{1}{r} \int_{B_1} q(x) \Delta u(rx+y) dx.$$

*Proof.* Let  $\tilde{u}_r(x) = u(rx + y)/r^2$ . From Lemmas 2.8 and 2.9 we obtain

$$\begin{aligned}
\int_{\partial B_1} q(x) \frac{d}{dr} Q_y(u, r)(x) d\mathcal{H}^{n-1} &= \frac{1}{r} \int_{\partial B_1} q(x) Q_0 \left( \frac{rx \cdot \nabla u(rx + y) - 2u(rx + y)}{r^2}, 1 \right) d\mathcal{H}^{n-1} \\
&= \frac{1}{r} \int_{\partial B_1} q(x) Q_0 (x \cdot \nabla \tilde{u}_r(x) - 2\tilde{u}_r(x), 1) d\mathcal{H}^{n-1} \\
&= \frac{1}{r} \int_{\partial B_1} q(x) (x \cdot \nabla \tilde{u}_r(x) - 2\tilde{u}_r(x)) d\mathcal{H}^{n-1} \\
&= \frac{1}{r} \int_{B_1} q(x) \Delta \tilde{u}_r(x) dx = \frac{1}{r} \int_{B_1} q(x) \Delta u(rx + y) dx.
\end{aligned}$$

□

**Lemma 2.11.** *For  $u \in W^{2,p}(B_1(y))$  with  $p$  large enough and  $0 < r \leq 1$ ,*

$$\frac{d}{dr} \int_{\partial B_1} Q_y^2(u, r) d\mathcal{H}^{n-1} = \frac{2}{r} \int_{B_1} Q_y(u, r) \Delta u(rx + y) dx.$$

*Proof.* By Lemmas 2.8, 2.10 we get

$$\begin{aligned}
\frac{d}{dr} \int_{\partial B_1} Q_y^2(u, r) d\mathcal{H}^{n-1} &= 2 \int_{\partial B_1} Q_y(u, r) \frac{d}{dr} Q_y(u, r) d\mathcal{H}^{n-1} \\
&= \frac{2}{r} \int_{B_1} Q_y(u, r) \Delta u(rx + y) dx.
\end{aligned}$$

□

**Lemma 2.12.** *Let  $u$  be a solution of (1) and  $y \in B_{1/2}$ . For  $0 < r < 1/2$  consider*

$$\begin{aligned}
u_r(x) &:= \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - \Pi_y(u, r), \\
v_r(x) &:= \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - Q_y(u, r).
\end{aligned}$$

*Then*

- i.  $u_r - v_r$  is bounded in  $C^\infty$ , uniformly in  $r$ ;
- ii. the family  $\{v_r\}$  is bounded in  $C^{1,\alpha}(B_1) \cap W^{2,p}(B_1)$ , for every  $0 < \alpha < 1$  and  $p > 1$ .

*Proof.* i. For each  $r$ , the difference  $u_r - v_r = Q_y(u, r) - \Pi_y(u, r)$  is a second order harmonic polynomial. Therefore, it suffices to show that  $L^\infty$  norm of that difference admits a bound independent of  $r$ . Note that

$$\begin{aligned} u_r - v_r &= Q_y(u, r) - \Pi_y(u, r) \\ &= Q_0 \left( \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - \Pi_y(u, r), 1 \right) = Q_0(u_r, 1). \end{aligned}$$

Hence,

$$\sup_r \sup_{B_1} |Q_0(u_r, 1)| \leq C \sup_r \sup_{B_1} |u_r| < \infty.$$

ii. Lemma 2.2 implies that  $\{u_r\}_{r>0}$  is bounded in  $C^{1,\alpha}(B_1) \cap W^{2,p}(B_1)$  for every  $\alpha < 1$  and  $p > 1$ . Hence, the result follows from i.  $\square$

### 3 $C^{1,1}$ regularity: general case

In this section we utilize the previous technical tools and prove  $C^{1,1}$  regularity provided that  $f = f(x, t)$  satisfies assumption **A**:

**Assumption A.**

(i)

$$|f(x, t_2) - f(x, t_1)| \leq h(x)\omega(|t_2 - t_1|),$$

where  $h \in L^\infty(B_1)$  and

$$\int_0^\epsilon \frac{\omega(t)}{t} dt < \infty,$$

for some  $\epsilon > 0$ ;

(ii) The Newtonian potential of  $x \mapsto f(x, t)$  is  $C^{1,1}$  locally uniformly in  $t$ : for  $v_t := f(\cdot, t) * N$  where  $N$  is the Newtonian potential,

$$\sup_{a \leq t \leq b} \|D^2 v_t\|_{L^\infty(B_1)} < \infty, \quad \text{for all } a, b \in \mathbb{R}.$$

**Proof of Theorem 1.1.** Let  $y \in B_{1/2}$  and  $v = v_{u(y)} = f(x, u(y)) * N$ . Note that if

$$u_r(x) = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - \Pi_y(u, r),$$

then

$$\Pi_y(u, r/2) - \Pi_y(u, r) = \Pi_y(u_r, 1/2) - \Pi_y(u_r, 1) = \Pi_y(u_r, 1/2).$$

Using this identity, Lemma 2.3, and Lemma 2.5

$$\begin{aligned}
& \|\Pi_y(u, r/2) - \Pi_y(u, r) - \Pi_y(v, r/2) + \Pi_y(v, r)\|_{L^\infty(B_1)} \\
&= \|\Pi_y(u_r, 1/2) - \Pi_y(v_r, 1/2) - \Pi_y(u_r, 1) + \Pi_y(v_r, 1)\|_{L^\infty(B_1)} \\
&= \|\Pi_y(u_r - v_r, 1/2) - \Pi_y(u_r - v_r, 1)\|_{L^\infty(B_1)} \\
&\leq C\|\Delta u_r - \Delta v_r\|_{L^2(B_1)} \\
&= \|f(rx + y, u(rx + y)) - f(rx + y, u(y))\|_{L^2(B_1)} \\
&\leq C\omega\left(\sup_{B_r(y)} |u(x) - u(y)|\right) \leq C\omega\left(c(r + r^2 \log \frac{1}{r})\right) \leq C\omega(cr),
\end{aligned}$$

for  $r > 0$  sufficiently small ( $|\nabla u(y)|$  is controlled by  $\|u\|_{W^{2,p}(B_1)}$ ). Hence, for  $r_0 > 0$  small enough and  $y \in B_{1/2}$  we have

$$\begin{aligned}
& \|\Pi_y(u, r_0/2^j) - \Pi_y(u, r_0)\|_{L^\infty(B_1)} \\
&\leq \left\| \sum_{k=1}^j \Pi_y(v, r_0/2^k) - \Pi_y(v, r_0/2^{k-1}) \right\|_{L^\infty(B_1)} \\
&+ \sum_{k=1}^j \left\| \Pi_y(u, r_0/2^k) - \Pi_y(u, r_0/2^{k-1}) - \Pi_y(v, r_0/2^k) + \Pi_y(v, r_0/2^{k-1}) \right\|_{L^\infty(B_1)} \\
&\leq C\|D^2 v_{u(y)}\|_{L^\infty(B_1)} + C \sum_{k=1}^\infty \omega\left(\frac{cr}{2^{k-1}}\right) \leq \tilde{C}(\|D^2 v_{u(y)}\|_{L^\infty(B_1)} + 1) \\
&\leq \tilde{C}\left(\sup_{|s| \leq \sup |u|} \|D^2 v_s\|_{L^\infty(B_1)} + 1\right).
\end{aligned}$$

Thus

$$\|\Pi_y(u, r_0/2^j)\|_{L^\infty(B_1)} \leq \|\Pi_y(u, r_0)\|_{L^\infty(B_1)} + \tilde{C}(\|D^2 v_{u(y)}\|_{L^\infty} + 1). \quad (5)$$

We conclude via Lemma 2.6 and Lemma 2.4.  $\square$

*Remark 1.* To generate examples, consider  $f(x, t) = \phi(x)\psi(t)$ . If  $\phi \in L^\infty$  and  $\psi$  is Dini, then  $f$  satisfies condition (i). If  $\phi * N$  is  $C^{1,1}$  and  $\psi$  is locally bounded, then  $f$  satisfies (ii). Thus if  $\phi * N$  is  $C^{1,1}$  and  $\psi$  is Dini, then  $f$  satisfies both conditions. In particular,  $f$  may be strictly weaker than Dini in the  $x$ -variable.

*Remark 2.* The projection  $Q_y$  has similar properties to  $\Pi_y$ . Consequently, if  $f$  satisfies assumption **A**, (5) holds for  $\Pi_y$  replaced by  $Q_y$ .

## 4 $C^{1,1}$ regularity: discontinuous case

The goal of this section is to investigate the optimal regularity for solutions of (1) with  $f$  having a jump discontinuity in the  $t$ -variable. This case may be viewed as a free boundary problem. The idea is to employ again an  $L^2$  projection operator.

## 4.1 Two-phase obstacle problem

Suppose  $f = f(x, u)$  has the form

$$f(x, u) = g_1(x, u)\chi_{\{u>0\}} + g_2(x, u)\chi_{\{u<0\}},$$

where  $g_1, g_2$  are continuous. We recall from the introduction that if  $f$  has a jump in  $u$  at the origin, then we assume it to be a positive jump:

**Assumption B.**  $g_1(x, 0) - g_2(x, 0) \geq \sigma_0$ ,  $x \in B_1$  for some  $\sigma_0 > 0$ .

*Remark 3.* In the unstable obstacle problem, i.e.  $g_1 = -1$ ,  $g_2 = 0$ , there exists a solution which is  $C^{1,\alpha}$  for any  $\alpha \in (0, 1)$  but not  $C^{1,1}$ .

Let  $\Gamma^0 := \Gamma \cap \{|\nabla u| = u = 0\}$  and  $\Gamma^1 := \Gamma \cap \{|\nabla u| \neq 0\}$ . Our main result provides optimal growth away from points with sufficiently small gradients.

**Theorem 4.1.** *Suppose  $g_1, g_2 \in C^0$  satisfy **B**. Then for all constants  $\theta, M > 0$  there exist  $r_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$  and  $C_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$  such that for any solution of (1) with  $\|u\|_{L^\infty(B_1)} \leq M$*

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C_0, \quad (6)$$

for all  $r \leq r_0$  and  $y \in B_{1/2} \cap \Gamma \cap \{|\nabla u(y)| < \theta r\}$ . Consequently, for the same choice of  $r$  and  $y$  we have that

$$\sup_{x \in B_r} |u(x + y) - x \cdot \nabla u(y)| \leq C_1 r^2, \quad (7)$$

for some constant  $C_1(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ .

The proof of the theorem is carried out in several steps. A crucial ingredient is the following monotonicity result.

**Lemma 4.2.** *Suppose  $g_1, g_2 \in C^0$  satisfy **B**. Then for all constants  $\theta, M > 0$  there exist  $\kappa_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$  and  $r_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$  such that for any solution  $u$  of (1) with  $\|u\|_{L^\infty(B_1)} \leq M$  if*

$$\|Q_y(u, r)\|_{L^2(\partial B_1)} \geq \kappa_0,$$

for some  $0 < r < r_0$  and  $y \in B_{1/2} \cap \Gamma \cap \{|\nabla u(y)| < \theta r\}$ , then

$$\frac{d}{dr} \int_{\partial B_1} Q_y^2(u, r) d\mathcal{H}^{n-1} > 0.$$

*Proof.* If the conclusion is not true, then there exist radii  $r_k \rightarrow 0$ , solutions  $u_k$  and points  $y_k \in B_{1/2} \cap \Gamma_k \cap \{|\nabla u_k(y_k)| < \theta r_k\}$  such that  $\|u_k\|_{L^\infty(B_1)} \leq M$ , and  $\|Q_{y_k}(u_k, r_k)\|_{L^2(\partial B_1)} \rightarrow \infty$ , and

$$\frac{d}{dr} \int_{\partial B_1} Q_{y_k}^2(u_k, r) d\mathcal{H}^{n-1} \Big|_{r=r_k} \leq 0.$$

Let

$$T_k := \|Q_{y_k}(u_k, r_k)\|_{L^2(\partial B_1)},$$

and consider the sequence

$$v_k(x) = \frac{u_k(r_k x + y_k) - r_k x \cdot \nabla u_k(y_k)}{r_k^2} - Q_{y_k}(u_k, r_k).$$

Without loss of generality we can assume that  $y_k \rightarrow y_0$  for some  $y_0 \in B_{1/2}$ . Lemma 2.2 implies the existence of a function  $v$  such that up to a subsequence

$$v_k(x) = \frac{u_k(r_k x + y_k) - r_k x \cdot \nabla u_k(y_k)}{r_k^2} - Q_{y_k}(u_k, r_k) \rightarrow v, \text{ in } C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n) \cap W_{\text{loc}}^{2,p}(\mathbb{R}^n).$$

Evidently,  $v(y_0) = |\nabla v(y_0)| = 0$ . Moreover, for  $q_k(x) := Q_{y_k}(u_k, r_k)/T_k$ , we can assume that up to a further subsequence,  $q_k \rightarrow q$  in  $C^\infty$  for some  $q \in \mathcal{P}_2$ . Note that

$$\begin{aligned} \Delta v_k(x) &= g_1(r_k x + y_k, u_k(r_k x + y_k)) \chi_{\{u_k(r_k x + y_k) > 0\}} \\ &\quad + g_2(r_k x + y_k, u_k(r_k x + y_k)) \chi_{\{u_k(r_k x + y_k) < 0\}} \end{aligned}$$

hence

$$\Delta v_k \rightarrow \Delta v = g_1(y_0, 0) \chi_{\{q(x) > 0\}} + g_2(y_0, 0) \chi_{\{q(x) < 0\}}.$$

By Lemma 2.11,

$$\begin{aligned} 0 &\geq \frac{d}{dr} \int_{\partial B_1} Q_{y_k}^2(u_k, r) d\mathcal{H}^{n-1} \Big|_{r=r_k} = \frac{2}{r_k} \int_{B_1} Q_{y_k}(u_k, r_k) \Delta u_k(r_k x + y_k) dx \\ &= \frac{2T_k}{r_k} \int_{B_1} q_k(x) \Delta v_k(x) dx. \end{aligned}$$

Therefore

$$\int_{B_1} q_k(x) \Delta v_k(x) dx \leq 0.$$

On the other hand

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{B_1} q_k(x) \Delta v_k(x) dx &= \int_{B_1} q(x) (g_1(0, y_0) \chi_{\{q(x) > 0\}} + g_2(0, y_0) \chi_{\{q(x) < 0\}}) dx \\ &= (g_1(0, y_0) - g_2(0, y_0)) \int_{q(x) > 0} q(x) dx > 0, \end{aligned}$$

a contradiction.  $\square$

**Proof of Theorem 4.1.** Let  $\kappa_0$  and  $r_0$  be the constants from Lemma 4.2. Without loss of generality we can assume that  $r_0 \leq 1/4$ . From Lemmas 2.4 and 2.12 we have that

$$\|Q_y(u, r_0)\|_{L^2(\partial B_1)} \leq C \log \frac{1}{r_0},$$

for all  $y \in B_{1/2}$ , where  $C = C(M, \|g_1\|_\infty, \|g_2\|_\infty, n)$  is a constant. Take

$$C_0 = \max \left( k_0, 2C \log \frac{1}{r_0} \right).$$

We claim that

$$\|Q_y(u, r)\|_{L^2(\partial B_1)} \leq C_0,$$

for  $r \leq r_0$  and  $y \in B_{1/2} \cap \Gamma \cap \{|\nabla u(y)| < \theta r\}$ . Let us fix  $y$  such that  $|\nabla u(y)| \leq \theta r_0$  and consider

$$T_y(r) := \|Q_y(u, r)\|_{L^2(\partial B_1)}$$

as a function of  $r$  on the interval  $|\nabla u(y)|/\theta \leq r \leq r_0$ . Let

$$e := \inf\{r \text{ s.t. } T_y(r) \leq C_0\}. \quad (8)$$

We have that  $T_y(r_0) < C_0$ , so  $|\nabla u(y)|/\theta \leq e < r_0$ . If  $e > |\nabla u(y)|/\theta$  then  $T_y(e) = C_0$  and by Lemma 4.2 we have that  $T'_y(e) > 0$ , so  $T_y(r) < C_0$  for  $e - \varepsilon < r < e$  which contradicts (8).

Therefore,  $e = |\nabla u(y)|/\theta$  and  $T_y(r) \leq C_0$  for all  $|\nabla u(y)|/\theta \leq r \leq r_0$  which proves (6).

Inequality (7) follows from Lemmas 2.2 and 2.12.  $\square$

Theorem 4.1 implies  $C^{1,1}$  regularity away from  $\Gamma^1$  in the case the coefficients  $g_i$  are regular enough to provide  $C^{1,1}$  solutions away from the free boundary, i.e. Theorem 1.2.

*Remark 4.* Note that **A** is the condition given in Theorem 1.1. If  $g_i$  only depend on  $x$ , then this reduces to the assumption that the Newtonian potential of  $g_i$  is  $C^{1,1}$ , which is sharp.

**Proof of Theorem 1.2.** Suppose **A** and **B** hold. We show that for every  $\delta > 0$  there exists  $C_\delta > 0$  such that for all  $y \in B_{1/2}(0)$  such that  $\text{dist}(y, \Gamma^1) \geq \delta$ , there exists  $r_y > 0$  such that

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C_\delta, \quad (9)$$

for  $r \leq r_y$ .

Consequently,

$$|u(x) - u(y) - \nabla u(y)(x - y)| \leq \tilde{C}_\delta |x - y|^2 \quad (10)$$

for  $|x - y| \leq r_y$ ,  $y \in B_{1/2}(0)$  and  $\text{dist}(y, \Gamma^1) \geq \delta$ ; this readily yields the desired result.

Note that (10) follows from (9) via Lemmas 2.2 and 2.12.

Without loss of generality assume that  $\delta \leq r_0$ , where  $r_0 > 0$  is the constant from Theorem 4.1. For every  $y \in B_{1/2}(0)$  consider the ball  $B_{\delta/2}(y)$ . Then there are two possibilities.

i.  $B_{\delta/2}(y) \cap \Gamma^0 = \emptyset$ .

In this case  $B_{\delta/2} \cap \Gamma = \emptyset$ , hence  $u$  satisfies the equation

$$\Delta u = g_i(x, u)$$

in  $B_{\delta/2}(y)$  for  $i = 1$  or  $i = 2$ . Inequality (5) in the Theorem 1.1 assumption **A** yields

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C \log \frac{4}{\delta} + C(\|D^2 v_{u(y)}^i\|_{\infty} + 1),$$

for  $r \leq \delta/4$ .

ii.  $B_{\delta/2}(y) \cap \Gamma^0 \neq \emptyset$ .

Let  $w \in \Gamma^0$  be such that  $d := |y - w| = \text{dist}(y, \Gamma_0)$ . We have that  $d \leq \delta/2$ . As before, assumption **A** yields

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} + C(\|D^2 v_{u(y)}^i\|_{\infty} + 1),$$

for  $r \leq d/2$ . From Theorem 4.1 we have that

$$\left| u \left( y + \frac{d}{2} z \right) \right| \leq C \left| y + \frac{d}{2} z - w \right|^2 \leq C d^2,$$

for all  $|z| \leq 1$  because  $d \leq \delta/2 \leq r_0$ . On the other hand

$$\begin{aligned} Q_y(u, d/2) &= \text{Proj}_{\mathcal{P}_2} \left( \frac{u \left( y + \frac{d}{2} z \right) - \frac{d}{2} z \cdot \nabla u(y) - u(y)}{d^2/4} \right) \\ &= \text{Proj}_{\mathcal{P}_2} \left( \frac{u \left( y + \frac{d}{2} z \right)}{d^2/4} \right), \end{aligned}$$

where  $\text{Proj}_{\mathcal{P}_2}$  is the  $L^2(\partial B_1(0))$  projection on the space  $\mathcal{P}_2$ . We have used the fact that the projection of a linear function is 0. Hence

$$\|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} \leq \left\| \frac{u \left( y + \frac{d}{2} z \right)}{d^2/4} \right\|_{L^2(\partial B_1(0))} \leq C,$$

which yields

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C + C(\|D^2 v_{u(y)}^i\|_{\infty} + 1),$$

for  $r \leq d/2$ .

The proof is now complete.  $\square$



Lastly we point out that if the coefficients  $g_i$  are regular enough to provide  $C^{1,1}$  solutions at points where the gradient does not vanish, then we obtain full interior  $C^{1,1}$  regularity.

**Assumption C.** For any  $M > 0$  there exist  $\theta_0(M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$  and  $C_3(M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$  such that for all  $z \in B_{1/2}$  any solution of

$$\begin{cases} \Delta v = g_1(x, v)\chi_{v>0} + g_2(x, v)\chi_{v<0}, & x \in B_{1/2}(z); \\ |v(x)| \leq M, & x \in B_{1/2}(z); \\ v(z) = 0, & 0 < |\nabla v(z)| \leq \theta_0/4; \\ v|_{\partial B_r(z)} \text{ continuous,} \end{cases}$$

admits a bound

$$\|D^2 v\|_{L^\infty(B_{|\nabla v(z)|/\theta_0}(z))} \leq C_3.$$

*Remark 5.* A sufficient condition which ensures **C** is that  $g_i$  are Hölder continuous, see [LSE09, Proposition 2.6] and [ADN64, Theorem 9.3]. The idea being that at such points, the set  $\{u = 0\}$  is locally  $C^{1,\alpha}$  (via the implicit function theorem) and one may thereby reduce the problem to a classical PDE for which up to the boundary estimates are known.

Theorem 4.1 and **C** imply Theorem 1.3.

**Proof of Theorem 1.3.** By Lemmas 2.12 and 2.6 the assertion follows if we show that there exist  $\rho_0, C > 0$  such that for every  $y \in B_{\rho_0}(0)$  there exists  $r_y > 0$  such that

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C \tag{11}$$

for  $0 < r \leq r_y$ .

Let  $\rho_0$  be such that  $|\nabla u(y)| \leq \theta_0$  for  $y \in B_{\rho_0}(0)$ , where  $\theta_0$  is the constant from assumption **C** (we can do this because  $u$  is  $C^{1,\alpha}$  and  $0 \in \Gamma^0$ ). For  $y \in B_{\rho_0}(0)$  let  $d := \text{dist}(y, \Gamma)$  and let  $w \in \Gamma$  be such that  $d = |y - w|$ .

From Corollary 1.2 we can assume that  $2d < r_0$ . One of the following cases is possible.

i.  $d = 0, y \in \Gamma^0$ .

In this case we have that (11) holds for  $r \leq r_0$  by Theorem 4.1.

ii.  $d = 0, y \in \Gamma^1$ .

Here, (11) follows from the assumption **C**.

iii.  $d > 0, w \in \Gamma^0$ .

$u$  solves  $\Delta u = g_i(x, u)$  in  $B_{d/2}(y)$  for  $i = 1$  or  $i = 2$ . Then, by the analysis similar to the one in Corollary 1.2 we get that (11) holds for  $r \leq d/2$ .

iv.  $d > 0, w \in \Gamma^1$ .

From Theorem 4.1 we have that

$$|u(z+w) - z \cdot \nabla u(w)| \leq C_1 |z|^2 \quad (12)$$

for  $|\nabla u(w)|/\theta_0 \leq |z| \leq r_0$ . On the other hand by assumption **C** we obtain that (12) holds for  $|z| \leq |\nabla u(w)|/\theta_0$ . Hence, (12) holds for all  $z$  such that  $|z| \leq r_0$ .

By assumption **A** we have that

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} + C(\|D^2 v_{u(y)}^i\|_\infty + 1),$$

for  $r \leq d/2$ .

Furthermore,

$$\begin{aligned} Q_y(u, d/2) &= \text{Proj}_{\mathcal{P}_2} \left( \frac{u(y + \frac{d}{2}z) - \frac{d}{2}z \cdot \nabla u(y) - u(y)}{d^2/4} \right) \\ &= \text{Proj}_{\mathcal{P}_2} \left( \frac{u(y + \frac{d}{2}z) - (y + \frac{d}{2}z - w) \cdot \nabla u(w)}{d^2/4} \right). \end{aligned}$$

Hence from (12) we get

$$\begin{aligned} \|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} &\leq \left\| \frac{u(y + \frac{d}{2}z) - (y + \frac{d}{2}z - w) \cdot \nabla u(w)}{d^2/4} \right\|_{L^2(\partial B_1(0))} \\ &\leq C, \end{aligned}$$

which yields

$$\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C + C(\|D^2 v_{u(y)}^i\|_\infty + 1),$$

for  $r \leq d/2$ .

□

The previous analysis applies to the following example.

**Example.** Let  $g_i(x, u) = \lambda_i(x)$  for  $i = 1, 2$ , where  $\lambda_i$  are such that

- i.  $\lambda_1(x) - \lambda_2(x) \geq \sigma_0 > 0$  for all  $x \in B_1$ ;
- ii.  $\lambda_1(x), \lambda_2(x)$  are Hölder continuous.

We recall from the introduction that under the stronger assumption  $\inf_{B_1} \lambda_1 > 0, \inf_{B_1} \lambda_2 > 0$ , this problem is studied in [LSE09] and the optimal interior  $C^{1,1}$  regularity is established. The authors use a different approach based on monotonicity formulas and an analysis of global solutions via a blow-up procedure.

## 4.2 No-sign obstacle problem

Here we observe that assumption **A** implies that the solutions of (3) are in  $C^{1,1}(B_{1/2})$ . This theorem was proven in [ALS13] (Theorem 1.2) for the case when  $g(x, t)$  depends only on  $x$ . Under assumption **A**, appropriate modifications of the proof in [ALS13] work also for the general case; since the arguments are similar, we provide only a sketch of the proof and highlight the differences.

**Sketch of the proof of Theorem 1.4.** Let  $\tilde{\Gamma} := \{y \text{ s.t. } u(y) = |\nabla u(y)| = 0\}$ . For  $r > 0$  let  $\Lambda_r := \{x \in B_1 \text{ s.t. } u(rx) = 0\}$  and  $\lambda_r := |\Lambda_r|$ .

The proof of Theorem 1.2 in [ALS13] consists of the following ingredients.

- Interior  $C^{1,1}$  estimate
- Quadratic growth away from the free boundary
- [ALS13, Proposition 5.1]

Let us recall that the interior  $C^{1,1}$  estimate is the inequality

$$\|u\|_{C^{1,1}(B_{d/2})} \leq C \left( \|g\|_{L^\infty(B_d)} + \frac{\|u\|_{L^\infty(B_d)}}{d^2} \right), \quad (13)$$

where  $\Delta u(x) = g(x)$  for  $x \in B_d$  and the Newtonian potential of  $g$  is  $C^{1,1}$ . This estimate is purely a consequence of  $g$  having a  $C^{1,1}$  Newtonian potential.

Quadratic growth away from the free boundary is a bound

$$|u(x)| \leq C \text{dist}(x, \tilde{\Gamma})^2. \quad (14)$$

The first observation in [ALS13] is that if  $g(x, t) = g(x)$  has a  $C^{1,1}$  Newtonian potential, then (14) and (13) yield  $C^{1,1}$  regularity for the solution. Indeed, “far” from the free boundary, the solution  $u$  solves the equation  $\Delta u = g(x)$  and is locally  $C^{1,1}$  by assumption. For points close to the free boundary,  $u$  solves the same equation but now on a small ball centered at the point of interest and touching the free boundary. At this point one invokes (14) and by (13) obtains that the  $C^{1,1}$  bound does not blow up close to the free boundary (see Lemma 4.1 in [ALS13]).

To prove (14), the authors prove in Proposition 5.1 [ALS13] that if the projection  $\Pi_y(u, r)$  (for some  $y \in \tilde{\Gamma}$ ) is large enough then the density  $\lambda_r$  of the coincidence set diminishes at an exponential rate. On the other hand, if  $\lambda_r$  diminishes in an exponential rate,  $\Pi_y(u, r)$  has to be bounded. Consequently, by invoking Lemma 2.2 one obtains (14).

Now let  $g$  satisfy **A**.

- Interior  $C^{1,1}$  estimate

In the general case, (13) is replaced by

$$\|Q_y(u, s)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u, r)\|_{L^2(\partial B_1(0))} + C(\|D^2 v_{u(y)}\|_\infty + 1), \quad (15)$$

where  $0 < s < r < d$ ,  $\Delta v_{u(y)} = g(x, u(y))$  and  $\Delta u = f(x, u)$  in  $B_d(y)$ . Estimate (15) is purely a consequence of assumption **A** (see (5) in the proof of Theorem 1.1).

- [ALS13, Proposition 5.1]

In this proposition, it is shown that there exists  $C$  such that if  $\Pi_y(u, r) \geq C$  then

$$\lambda_{r/2}^{1/2} \leq \frac{\tilde{C}}{\|\Pi_y(u, r)\|_{L^\infty(B_1)}} \lambda_r^{1/2} \quad (16)$$

for some  $\tilde{C} > 0$ . The inequality is obtained by the decomposition

$$\frac{u(rx + y)}{r^2} = \Pi_y(u, r) + h_r + w_r,$$

where  $h_r, w_r$  are such that

$$\begin{cases} \Delta h_r = -g(rx + y)\chi_{\Lambda_r} & \text{in } B_1, \\ h_r = 0 & \text{on } \partial B_1, \end{cases}$$

and

$$\begin{cases} \Delta w_r = g(rx + y) & \text{in } B_1, \\ w_r = \frac{u(rx + y)}{r^2} - \Pi_y(u, r) & \text{on } \partial B_1. \end{cases}$$

The authors show that

$$\begin{aligned} \|D^2 h_r\|_{L^2(B_{1/2})} &\leq C \|g\|_{L^\infty} \|\chi_{\Lambda_r}\|_{L^2(B_1)}, \\ \|D^2 w_r\|_{L^\infty(B_{1/2})} &\leq C (\|g\|_{L^\infty} + \|u\|_{L^\infty(B_1)}). \end{aligned} \quad (17)$$

In the general case one may consider the decomposition

$$\frac{u(rx + y)}{r^2} = Q_y(u, r) + h_r + w_r + z_r,$$

where  $h_r, w_r, z_r$  are such that

$$\begin{cases} \Delta h_r = -g(rx + y, 0)\chi_{\Lambda_r} & \text{in } B_1, \\ h_r = 0 & \text{on } \partial B_1, \end{cases}$$

and

$$\begin{cases} \Delta w_r = g(rx + y, 0) & \text{in } B_1, \\ w_r = \frac{u(rx + y)}{r^2} - Q_y(u, r) & \text{on } \partial B_1, \end{cases}$$

and

$$\begin{cases} \Delta z_r = (g(rx + y, u(rx + y)) - g(rx + y, 0))\chi_{B_1 \setminus \Lambda_r} & \text{in } B_1, \\ z_r = 0 & \text{on } \partial B_1. \end{cases}$$

Evidently, estimates (17) are still valid. Additionally, we have

$$\|D^2 z_r\|_{L^2(B_{1/2})} \leq C \|\Delta z_r\|_{L^2(B_1)} \leq C \omega(r^2 \log \frac{1}{r}), \quad (18)$$

since  $g(x, t)$  is uniformly Dini in  $t$ .

Combining (17) and (18) and arguing as in [ALS13] one obtains the existence of  $C > 0$  such that

$$\lambda_{r/2}^{1/2} \leq \frac{\tilde{C}}{\|Q_y(u, r)\|_{L^2(\partial B_1)}} \lambda_r^{1/2} + \omega \left( r^2 \log \frac{1}{r} \right), \quad (19)$$

whenever  $\|Q_y(u, r)\|_{L^2(\partial B_1)} \geq C$ .

- Quadratic growth away from the free boundary

In [ALS13], the norms of  $\Pi_y(u, r/2^k)$ ,  $k \geq 1$  are estimated in terms of the sum  $\sum_{j=0}^{\infty} \lambda_{r/2^j}$ . If the norms of projections are unbounded, one obtain estimate (16) which implies convergence of the previous sum and hence boundedness of the projections. This is a contradiction.

Similarly, in the general case the norms of  $Q_y(u, r/2^k)$ ,  $k \geq 1$  can be estimated by

$$\sum_{j=0}^{\infty} \lambda_{r/2^j} + \sum_{j=0}^{\infty} \omega \left( \left( \frac{r}{2^k} \right)^2 \log \frac{2^k}{r^2} \right).$$

Inequality (19) and Dini continuity imply

$$\sum_{j=0}^{\infty} \omega \left( \left( \frac{r}{2^k} \right)^2 \log \frac{2^k}{r^2} \right), \quad \sum_{j=0}^{\infty} \lambda_{r/2^j} < \infty,$$

if the norms of projections are unbounded. Furthermore, one completes the proof of the quadratic growth as in [ALS13].

To verify that the above ingredients imply  $C^{1,1}$  regularity, we split the analysis into two cases. If we are “far” from the free boundary,  $u$  locally solves  $\Delta u = g(x, u)$  so by Theorem 3.1  $u$  is  $C^{1,1}$ . If we are close to the free boundary then  $u$  solves  $\Delta u = g(x, u)$  in a small ball  $B_d(y)$  that touches the free boundary. We invoke (15) for  $0 < s < r = d/2$  and the quadratic growth to obtain

$$\begin{aligned} \|Q_y(u, s)\|_{L^2(\partial B_1(0))} &\leq \|Q_y(u, d/2)\|_{L^2(\partial B_1)} + C(\|D^2 v_{u(y)}\|_{\infty} + 1) \\ &\leq C \left\| \frac{u(y + d/2x)}{d^2/4} \right\|_{L^2(\partial B_1)} + C(\|D^2 v_{u(y)}\|_{\infty} + 1) \\ &\leq C + C(\|D^2 v_{u(y)}\|_{\infty} + 1). \end{aligned}$$

for  $s \leq d/2$ .

So there exists a constant  $C$  such that for all  $y \in B_{1/2}$  there exist radii  $r_j(y) \rightarrow 0$  such that

$$Q_y(u, r_j(y)) \leq C.$$

We conclude via Lemma 2.6.

□

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